THE NUMBER OF PARKING FUNCTIONS WITH CENTER OF A GIVEN LENGTH

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Abstract. Let $1 \leq r \leq n$ and suppose that, when the *Depth-first Search Algorithm* is applied to a given rooted labeled tree on $n+1$ vertices, exactly r vertices are visited before backtracking. Let R be the set of trees with this property. We count the number of elements of R.

For this purpose, we first consider a bijection, due to Perkinson, Yang and Yu, that maps R onto the set of parking function with *center* (defined by the authors in a previous article) of size r. A second bijection maps this set onto the set of parking functions with *run* r, a property that we introduce here. We then prove that the number of length n parking functions with a given run is the number of length n *rook words* (defined by Leven, Rhoades and Wilson) with the same run. This is done by counting related lattice paths in a ladder-shaped region. We finally count the number of length n *rook words* with run r, which is the answer to our initial question.

1. Introduction

Let T_n be the set of rooted labeled trees on the set of vertices $\{0, 1, \ldots, n\}$ with root $r = 0$, and let $T \in \mathsf{T}_n$. Suppose that the *Depth-first Search Algorithm* (DFS) is applied to T by starting at r and by visiting at each vertex the unvisited neighbor of highest label. If T is not a path with endpoint r, at a certain moment the algorithm will backtrack. In this paper we are concerned with the number of vertices that are visited before this happens.

More precisely, let $\mathbf{v} = \mathbf{v}(T) = (v_1, \ldots, v_k)$ be the ordered set of vertices different from the root that are visited *before backtracking*, and let $\operatorname{arm}(T) = k$ be the length of $\mathbf{v}$. We evaluate explicitly the enumerator

$$\mathcal{ALT}_n(t) = \sum_{T \in \mathsf{T}_n} t^{\operatorname{arm}(T)}\,.$$

It is well-known that $|\mathsf{T}_n| = (n+1)^{n-1} = |\mathsf{PF}_n|$, where PF_n is the set of *parking functions of length* n, consisting of the n-tuples $\mathbf{a} = (a_1, \ldots, a_n)$ of positive integers such that the ith entry *in ascending order* is always at most $i \in [n] := \{1, \ldots, n\}$.

Given any $\mathbf{a} \in [n]^n$, which we denote either as a word, $\mathbf{a} = a_1 \cdots a_n$, or as a function, $\mathbf{a} \colon [n] \to [n]$ such that $\mathbf{a}(i) = a_i$,

- let $z(\mathbf{a})$ be the maximum k for which there exist $1 \leq i_1 < \cdots < i_k \leq n$ such that $a_{i_1} \leq 1, \ldots, a_{i_k} \leq k$;
- let $\operatorname{run}(\mathbf{a})$ be the maximum k for which $[k] \subseteq \{a_1, \ldots, a_n\}$;
- let $\mathbf{a}$ be a *rook word* if $a_1 \leq \operatorname{run}(\mathbf{a})$, and let RW_n be the set of (length n) rook words.

This work was partially supported by CMUP (UID/MAT/00144/2019) and CIDMA (UID/MAT/04106/2019), which are funded by FCT (Portugal) with national (ME) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

The purpose of this paper is to prove that

$$\sum_{T\in \mathsf{T}_n} t^{\mathrm{arm}(T)} = \sum_{\mathbf{a}\in \mathsf{PF}_n} t^{z(\mathbf{a})} = \sum_{\mathbf{a}\in \mathsf{PF}_n} t^{\mathrm{run}(\mathbf{a})} = \sum_{\mathbf{a}\in \mathsf{RW}_n} t^{\mathrm{run}(\mathbf{a})}\,,$$

with which we evaluate this function.

Parking functions are important combinatorial structures with several connections to other areas of mathematics (see e.g. Haglund [6] and the excellent survey by Yan [15]). In particular, starting with Kreweras [9], various bijections between trees on $n+1$ vertices and parking functions of length n were defined where the reversed sum enumerator for parking functions is the counterpart to the inversion enumerator for trees [7, 5, 12, 15]. In a recent paper, Perkinson, Yang and Yu [11] constructed a very general algorithm that gives us as a particular case a new bijection with this property.

We show in Section 2 (cf. [3]) that, for this bijection, the counterpart of the statistics $\mathrm{arm}(T)$ is $z(\mathbf{a}) = |Z(\mathbf{a})|$, where $Z(\mathbf{a})$ is *the center of* $\mathbf{a}$ defined in [2]. We recall that, for any $a \in [n]^n$, the center of $\mathbf{a}$ is the largest subset $X = \{x_1, \ldots, x_\ell\}$ of $[n]$ such that $1 \le x_1 < \cdots < x_\ell \le n$ and $a_{x_i} \le i$ for every $i \in [\ell]$. Namely, we prove that if $T \mapsto a$ under our bijection and $\mathbf{v}(T) = (v_1, \ldots, v_k)$, then the set $Z(\mathbf{a})$ is exactly $\{v_1, \ldots, v_k\}$. Hence, we obtain the following result, if we consider the enumerator

$$\mathcal{ZPF}_n(t) = \sum_{\mathbf{a}\in \mathsf{PF}_n} t^{z(\mathbf{a})}\,,$$

Theorem 2.1. *For every $n \in \mathbb{N}$,*

$$\mathcal{ALT}_n(t) = \mathcal{ZPF}_n(t)\,.$$

The evaluation of this new enumerator was indeed part of our initial twofold purpose for its role in the theory of parking functions, described as follows. Consider the Shi arrangement, formed by all the hyperplanes defined in $\mathbb{R}^n$ by equations of the form $x_i - x_j = 0$ and of the form $x_i - x_j = 1$, where $1 \le i < j \le n$. Let R_0 be the chamber of the arrangement consisting of the intersection of all the open slabs defined by the condition $0 < x_i - x_j < 1$. Pak and Stanley [13] defined a bijective labeling of the chambers of this arrangement by parking functions, in which R_0 is labeled with the parking function $(1, \ldots, 1)$ and, along a shortest path from R_0 to any other chamber, for any crossed hyperplane of the form $x_i - x_j = 0$ the jth coordinate of the label is increased by one, and for any crossed plane of the form $x_i - x_j = 1$ it is the ith coordinate that is increased by one. The bijection is defined from chambers (represented by permutations of $[n]$ decorated with arcs following certain rules) to parking functions. See [4] for a very general perspective of this bijection. It is from the *center* of any parking function that we may recover the chamber labeled by it in the Pak-Stanley labeling [2].

For example, consider the region $\mathcal{R}$ of the Shi arrangement in $\mathbb{R}^9$ defined by

$$\begin{aligned}
&x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5\,,\\
&x_8 + 1 > x_7,\ x_3 + 1 > x_2,\ x_7 + 1 > x_5\,,\\
&x_4 + 1 < x_1,\ x_6 + 1 < x_5\,.
\end{aligned}$$

Following Stanley [13], we represent $\mathcal{R}$ by the sequence of indices of coordinates in increasing order, decorated with non-nested arcs such that the integers $j > i$, with j on the left side of i, are covered with the same arc if and only if $x_j + 1 > x_i$ in $\mathcal{R}$. In the previous example, we have

$$8\,4\,3\,9\,6\,7\,1\,2\,5\,.$$

By the Pak-Stanley bijection, $\mathcal{R}$ is associated with the parking function

$$\mathbf{a} = 341183414\,.$$

To obtain $\mathcal{R}$ from $\mathbf{a}$, according to [2], note that the center of this parking function, $Z = \{3,4,6,7,8,9\}$, is formed by the elements of the first arc, and their positions in the permutation $\pi = 8\,4\,3\,9\,6\,7\,1\,2\,5$ can be determined step by step by knowing that $a_i - 1$ is the number of elements of Z less than i that are on the left side of i in π. In our example, since $a_4 = 1$, 4 must precede 3; since $a_6 = 3$, 6 must follow 43, etc. Graphically,

$$3 \preceq \pi \underset{x_4=1}{\longmapsto} 43 \preceq \pi \underset{x_6=3}{\longmapsto} 436 \preceq \pi \underset{x_7=4}{\longmapsto} 4367 \preceq \pi \underset{x_8=1}{\longmapsto} 84367 \preceq \pi \underset{x_9=4}{\longmapsto} 843967 \preceq \pi\ ,$$

where, given $\mathbf{a} = a_1 \cdots a_k$ and $\mathbf{b} = b_1 \cdots b_n$, $\mathbf{a} \preceq \mathbf{b}$ if $k \le n$ and whenever i precedes j in $\mathbf{a}$, i also precedes j in $\mathbf{b}$ [1]. This is the starting point for the recovery of $\mathcal{R}$ in [2], since it enables the replacement of the parking function by another one of shorter length, and so to proceed recursively.

We now consider a third statistic. Let, for $\mathbf{a} = (a_1, \ldots, a_n) \in [n]^n$ such that $1 \in \{a_1, \ldots, a_n\}$,

$$\mathrm{run}(\mathbf{a}) = \max\big\{ i \in [n] \mid [i] \subseteq \{a_1, \ldots, a_n\} \big\},$$

and let $\mathrm{run}(\mathbf{a}) = 0$ if $1 \notin \{a_1, \ldots, a_n\}$. We prove the following result in Section 3. Let

$$\mathcal{RPF}_n(t) = \sum_{\mathbf{a} \in \mathsf{PF}_n} t^{\mathrm{run}(\mathbf{a})}.$$

Theorem 3.3. *For every $n \in \mathbb{N}$,*

$$\mathcal{ZPF}_n(t) = \mathcal{RPF}_n(t)\,.$$

Now, consider the set RW_n of *rook words of length n* defined by Leven, Rhoades and Wilson [10], that is, the ordered sets $\mathbf{a} = (a_1, \ldots, a_n) \in [n]^n$ such that $a_1 \le \mathrm{run}(\mathbf{a})$. Let

$$\mathcal{RRW}_n(t) = \sum_{\mathbf{a} \in \mathsf{RW}_n} t^{\mathrm{run}(\mathbf{a})}$$

The key to our enumeration is developed in Section 4, where we prove the following result.
Theorem 4.11. *For every $n \in \mathbb{N}$,*

$$\mathcal{RPF}_n(t) = \mathcal{RRW}_n(t)\,.$$

In this case we do not consider all parking functions and all rook words at once. Instead, we only consider those for which the sets of elements with the same image are the same, i.e., with the same coimage (see Definition 4.7 below).

We count parking functions by counting nonnegative sequences that are componentwise bounded above by a given positive sequence. Based on results of independent interest we prove that their number is the number of rook words defined in the same way.

[1]We assume that, as functions, both $\mathbf{a}$ and $\mathbf{b}$ are injective.

Example 1.1. We consider in Table 1.1 the case where $n = 3$ and hence

$$\mathcal{ALT}_3(t) = \mathcal{ZPF}_3(t) = \mathcal{RPF}_3(t) = \mathcal{RRW}_3(t) = 4t + 6t^2 + 6t^3$$

by classifying the corresponding trees and parking functions according to the various statistics and the corresponding bijections. Note that for $n = 3$ rook words are parking functions and vice-versa, except that $311 \in \mathsf{PF}_3 \setminus \mathsf{RW}_3$ whereas $133 \in \mathsf{RW}_3 \setminus \mathsf{PF}_3$. But $\mathrm{run}(311) = \mathrm{run}(133) = 1$.

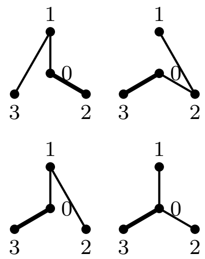
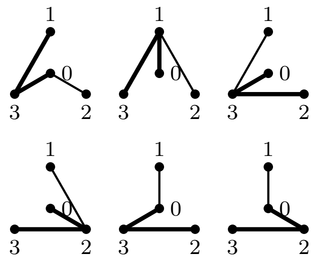
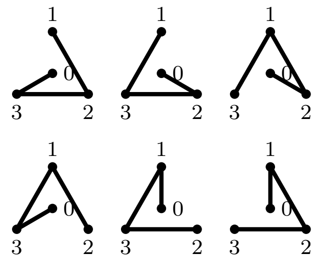
k	1	2	3
Trees T with $\mathrm{arm}(T) = k$			
Parking functions $\mathbf{a}$ with $z(\mathbf{a}) = k$	$2\underline{\mathbf{1}}3$ $22\underline{\mathbf{1}}$ $23\underline{\mathbf{1}}$ $32\underline{\mathbf{1}}$	$\underline{\mathbf{1}}3\underline{\mathbf{1}}$ $\underline{\mathbf{1}}3\underline{\mathbf{2}}$ $2\underline{\mathbf{11}}$ $2\underline{\mathbf{12}}$ $3\underline{\mathbf{11}}$ $3\underline{\mathbf{12}}$	$\underline{\mathbf{111}}$ $\underline{\mathbf{112}}$ $\underline{\mathbf{113}}$ $\underline{\mathbf{121}}$ $\underline{\mathbf{122}}$ $\underline{\mathbf{123}}$
Parking functions $\mathbf{a}$ with $\mathrm{run}(\mathbf{a}) = k$	$1\underline{\mathbf{1}}3$ $11\underline{\mathbf{1}}$ $13\underline{\mathbf{1}}$ $31\underline{\mathbf{1}}$	$\underline{\mathbf{2}}1\underline{\mathbf{1}}$ $\underline{\mathbf{1}}2\underline{\mathbf{2}}$ $2\underline{\mathbf{21}}$ $1\underline{\mathbf{12}}$ $1\underline{\mathbf{21}}$ $2\underline{\mathbf{12}}$	$\underline{\mathbf{321}}$ $\underline{\mathbf{231}}$ $\underline{\mathbf{213}}$ $\underline{\mathbf{312}}$ $\underline{\mathbf{132}}$ $\underline{\mathbf{123}}$
Rook words $\mathbf{a}$ with $\mathrm{run}(\mathbf{a}) = k$	$1\underline{\mathbf{1}}3$ $11\underline{\mathbf{1}}$ $13\underline{\mathbf{1}}$ $\underline{\mathbf{1}}33$	$\underline{\mathbf{2}}1\underline{\mathbf{1}}$ $\underline{\mathbf{1}}2\underline{\mathbf{2}}$ $2\underline{\mathbf{21}}$ $1\underline{\mathbf{12}}$ $1\underline{\mathbf{21}}$ $2\underline{\mathbf{12}}$	$\underline{\mathbf{321}}$ $\underline{\mathbf{231}}$ $\underline{\mathbf{213}}$ $\underline{\mathbf{312}}$ $\underline{\mathbf{132}}$ $\underline{\mathbf{123}}$

TABLE 1. The case where $n = 3$

Finally, by directly counting rook words with a given run, we are able to present in Section 5 an expression for all the (equal) previous enumerators.

Theorem 5.1. *For all integers* $1 \le r \le n$,

$$[t^r]\big(\mathcal{ALT}_n(t)\big) = r! \sum_{i_1+\cdots+i_r=n-r} (n-1)^{i_1}(n-2)^{i_2}\cdots(n-r)^{i_r}$$
$$= r \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} (n-1-j)^{n-1}\,.$$

It is perhaps worth noting that rook words were introduced in order to label the chambers of the *Ish arrangement*, defined in $\mathbb{R}^n$ by all the hyperplanes with equations of the form $x_i - x_j = 0$, as before, and of the form $x_1 - x_j = i$, where again $1 \le i < j \le n$. Several bijections, which preserve different properties, have been defined between the chambers of the Shi arrangement and the chambers of the Ish arrangement, particularly by Leven, Rhoades and Wilson using rook words [10]. In fact, our work here may be presented as another example of a general statement by Armstrong and Rhoades [1], saying that "The Ish arrangement is something of a 'toy model' for the Shi arrangement", in the sense that several properties are shared by both arrangements, but are easier to prove in case of the Ish arrangement than in the case of the Shi arrangement.

2. From labeled trees to parking functions: arms *vs.* centers

We reproduce here the algorithm (Algorithm 1, below) of Perkinson, Yang and Yu [11], in our case applied to the complete graph $G = K_{n+1}$ on V. When the algorithm takes as input a parking function $\mathbf{a}$ (or, more precisely, takes as input $\mathcal{P} = \mathbf{a} - 1\colon [n] \to \mathbb{N} \cup \{0\}$ such that $\mathcal{P}(i) = a_i - 1$) it returns the list `tree_edges` of edges of a spanning tree of G. This correspondence is a bijection.

Note that, in general, a spanning tree T of G is seen as a directed graph in which all paths lead away from the root. So, edge ij is written (i, j) if i is in the (unique) path from 0 to j (with no vertex between them). Note also that, by definition (cf. Line 7), if after running the algorithm both edges (i, j) and (i, k) belong to T and $j > k$ then DFS_FROM(j) has been called before DFS_FROM(k).

Algorithm 1 DFS**-burning algorithm**.

ALGORITHM

Input: $\mathcal{P}\colon V \setminus \{r\} \to \mathbb{N} \cup \{0\}$

1: `burnt_vertices` $= \{r\}$
2: `tree_edges` $= \{\,\}$
3: execute DFS_FROM(r)

Output: `burnt_vertices` and `tree_edges`

AUXILIARY FUNCTION

4: **function** DFS_FROM(i)
5:   **foreach** j adjacent to i in G, from largest numerical value to smallest **do**
6:     **if** $j \notin$ `burnt_vertices` **then**
7:       **if** $\mathcal{P}(j) = 0$ **then**
8:         append j to `burnt_vertices`
9:         append (i, j) to `tree_edges`
10:         DFS_FROM(j)
11:       **else**
12:         $\mathcal{P}(j) = \mathcal{P}(j) - 1$

Theorem 2.1. *For every $n \in \mathbb{N}$,*

$$\mathcal{ALT}_n(t) = \mathcal{ZPF}_n(t)$$

Proof. We show that there exist a bijection $\varphi\colon \mathsf{PF}_n \to \mathsf{T}_n$ such that, for every $\mathbf{a} \in \mathsf{PF}_n$, if $T = \varphi(\mathbf{a}) \in \mathsf{T}_n$, then $\operatorname{arm}(T) = z(\mathbf{a})$.

Let T be the tree given by Algorithm 1 with input $\mathcal{P} = \mathbf{a} - 1$ (we know this defines a bijection from PF_n to T_n by [11, Theorem 3]). Now, let ℓ be the first value of i where, when DFS_FROM(i) is called, $\mathcal{P}(j) > 0$ whenever $j \notin$ `burnt_vertices`. If this never occurs, let ℓ be the last vertex joined to `burnt_vertices`.

Let $B = \big(0 = v_0, \ldots, v_k = i\big) =$ `burnt_vertices` and $E =$ `tree_edges` at the end of the loop of DFS_FROM(i) (the end of Line 12) for $i = \ell$, and note that, by definition, $E = \big((v_0, v_1), \ldots, (v_{k-1}, v_k)\big)$. Hence, $\mathbf{v}(T) = (v_1, \ldots, v_k)$ and $\operatorname{arm}(\mathbf{a}) = k$.

Now, let $X = \{x_1, \ldots, x_k\} = \{v_1, \ldots, v_k\}$ with $x_1 < \cdots < x_k$.

We must prove that $X = Z(\mathbf{a})$, i.e., that:

(1) for every $m \in [k]$, $\mathbf{a}(x_m) \le m$;
(2) X is maximal for this property.

Clearly, if $x_{i_1} = v_1$, then $\mathbf{a}(x_{i_1}) \le i_1$ since, by definition, $v_1 = \max\big(\mathcal{P}^{-1}(\{0\})\big)$ and so $\mathbf{a}(x_{i_1}) = 1$. Now, suppose that the same holds true for $x_{i_2} = v_2, \ldots, x_{i_{\ell-1}} = v_{\ell-1}$, consider $x_{i_\ell} = v_\ell$ and note that, when DFS_FROM$(v_{\ell-1})$ is called, v_ℓ is the largest value of $j \notin \{v_0, v_1, \ldots, v_{\ell-1}\}$ with $\mathcal{P}(j) = 0$. Since $\mathcal{P}(v_\ell)$ has been reduced in earlier calls to

DFS_FROM(v_m) (at Line 12) exactly when $v_m < v_\ell$, since it is now zero, and since new additions to `burnt_vertices` will not decrease the order of v_ℓ in the corresponding set, $\mathbf{a}(x_{i_\ell}) \leq i_\ell$.

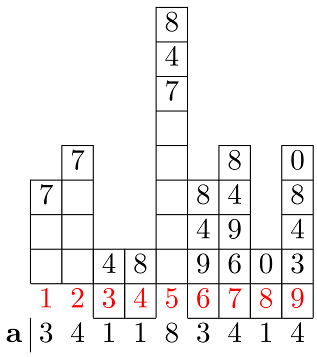
When finally DFS_FROM(v_k) is called, $\mathcal{P}(j) > 0$ for all $j \notin X$. In particular, if $Y \supsetneq X$, $Y = \{y_1, \ldots, y_{k'}\}$ with $y_1 < \cdots < y_{k'}$, j is the smallest element of $Y \setminus X$ and m is the number of elements of X less that y, then $j = y_m$ but $\mathbf{a}(j) > m$. $\square$

More precisely, if $\mathbf{v}(T) = (v_1, \ldots, v_k)$, then clearly

$$\mathbf{a}(v_i) = \left|\left\{t \in [i] \mid v_t \leq v_i\right\}\right| .$$

Compare with Definition 3.1 below.

Example 2.2. Let $\mathbf{a} = 341183414 \in \mathsf{PF}_9$. We apply Algorithm 1 to $\mathbf{a}$ by drawing a_j empty boxes for each $j \in [9]$ that are filled with i during the execution of DFS_FROM(i), at Line 14 and at Line 10. Below, DFS_FROM(i) has been called for, in this order, $i = 0, 8, 4, 3, 9, 6, 7$. Hence, at the moment, $i = 7$ and `burnt_vertices` $= (0, 8, 4, 3, 9, 6, 7)$. Since $\mathcal{P}(j) > 0$ for $j \notin$ `burnt_vertices` (i.e., for $j = 1, 2, 5$), $\ell = i = 7$, and so $\mathbf{v} = (8, 4, 3, 9, 6, 7)$.



3. Within parking functions: centers *vs.* runs

Definition 3.1. Consider, for a positive integer n and for a permutation $\mathbf{w} = (w_1, \ldots, w_n) \in \mathfrak{S}_n$,

$$f_{w_i} = \left|\left\{k \in [i] \ \mid \ w_k \leq w_i\right\}\right| \ , \ i = 1, \ldots, n \ ,$$

and

$$t_n(\mathbf{w}) = (f_1, \ldots, f_n) \in [1] \times [2] \times \cdots \times [n] \, .$$

According to [2], t_n is a bijection between $\mathfrak{S}_n$ and $[1] \times [2] \times \cdots \times [n]$.

Example 3.2. If $\mathbf{w} = 521634$, then $f_1 = f_{w_3} = 1$, $f_2 = f_{w_2} = 1$, $f_3 = f_{w_5} = 3$, $f_4 = f_{w_6} = 4$, $f_5 = f_{w_1} = 1$ and $f_6 = f_{w_4} = 4$. Hence $t(\mathbf{w}) = 113414 \in [1] \times \cdots \times [6]$.

Given $\mathbf{a} \in [n]^n$, let

$$\mathrm{Run}(\mathbf{a}) = \left\{ \max \ \mathbf{a}^{-1}(\{j\}) \mid 1 \leq j \leq \mathrm{run}(\mathbf{a}) \right\}$$

if $\mathrm{run}(\mathbf{a}) > 0$, and let $\mathrm{Run}(\mathbf{a}) = \varnothing$ if $\mathrm{run}(\mathbf{a}) = 0$. Then $|\,\mathrm{Run}(\mathbf{a})| = \mathrm{run}(\mathbf{a})$.

For $A \subseteq [n]$, let

$$Z_n{}^{-1}(A) = \left\{\mathbf{a} \in [n]^n \mid Z(\mathbf{a}) = A\right\}$$

and

$$\mathrm{Run}_n{}^{-1}(A) = \left\{\mathbf{a} \in [n]^n \mid \mathrm{Run}(\mathbf{a}) = A\right\}.$$

Now let $A = \{i_1, i_2, \ldots, i_k\} \neq \varnothing$ with $i_1 < i_2 < \cdots < i_k$ and take $i_0 = 0$ and $i_{k+1} = n+1$. Then $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in {Z_n}^{-1}(A)$ if and only if

- $(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) \in [1] \times [2] \times \cdots \times [k]$,
- $a_j \in \{\ell+1, \ldots, n\} = [n] \setminus [\ell]$, if $i_{\ell-1} < j < i_\ell$, with $\ell \in [k+1]$.

$$\begin{array}{cccccccccc}
\neq 1 & \boxed{a_{i_1}\leq 1} & \neq\begin{cases}1\\2\end{cases} & \boxed{a_{i_2}\leq 2} & \neq\begin{cases}1\\2\\3\end{cases} & & \neq\begin{cases}1\\ \vdots\\ k\end{cases} & \boxed{a_{i_k}\leq k} & \neq\begin{cases}1\\ \vdots\\ k+1\end{cases}\\
\underline{\qquad\qquad} & \underbrace{\qquad}_{i_1} & \underline{\qquad\qquad} & \underbrace{\qquad}_{i_2} & \underline{\qquad\qquad} & \cdots & \underline{\qquad\qquad} & \underbrace{\qquad}_{i_k} & \underline{\qquad\qquad}
\end{array}$$

On the other hand, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in {\mathrm{Run}_n}^{-1}(A)$ if and only if

- $(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) \in \mathfrak{S}_k$,
- $a_j \in [n] \setminus \{k+1, a_{i_1}, \ldots, a_{i_{\ell-1}}\}$, if $i_{\ell-1} < j < i_\ell$, with $\ell \in [k+1]$.

$$\begin{array}{cccccccccc}
\neq k+1 & \boxed{a_{i_1}} & \neq\begin{cases}k+1\\a_{i_1}\end{cases} & \boxed{a_{i_2}} & \neq\begin{cases}k+1\\a_{i_1}\\a_{i_2}\end{cases} & & \neq\begin{cases}k+1\\ \vdots\\ a_{i_{k-1}}\end{cases} & \boxed{a_{i_k}} & \neq\begin{cases}k+1\\ \vdots\\ a_{i_k}\end{cases}\\
\underline{\qquad\qquad} & \underbrace{\qquad}_{i_1} & \underline{\qquad\qquad} & \underbrace{\qquad}_{i_2} & \underline{\qquad\qquad} & \cdots & \underline{\qquad\qquad} & \underbrace{\qquad}_{i_k} & \underline{\qquad\qquad}
\end{array}$$

Clearly, both ${Z_n}^{-1}(A)$ and ${\mathrm{Run}_n}^{-1}(A)$ have size

$$k!(n-1)^{i_1-i_0-1}(n-2)^{i_2-i_1-1}\cdots(n-k-1)^{i_{k+1}-i_k-1}$$

if $|A| = k > 0$, and $(n-1)^n$ if $A = \varnothing$. We have the following result.

Theorem 3.3. *For every $n \in \mathbb{N}$,*

$$\mathcal{ZPF}_n(t) = \mathcal{RPF}_n(t)\,.$$

For completeness sake, we define two mappings $\Phi, \Psi : [n]^n \to [n]^n$ with the following properties.

Lemma 3.4.

(1) *For all $\mathbf{a} \in [n]^n$, $z(\mathbf{a}) = \mathrm{run}(\Phi(\mathbf{a}))$, $Z(\mathbf{a}) = \mathrm{Run}(\Phi(\mathbf{a}))$, $\mathrm{run}(\mathbf{a}) = z(\Psi(\mathbf{a}))$, and $\mathrm{Run}(\mathbf{a}) = Z(\Psi(\mathbf{a}))$,*
(2) *Φ and Ψ are bijections and $\Psi = \Phi^{-1}$,*
(3) *$\Phi(\mathsf{PF}_n) = \mathsf{PF}_n$.*

Definition 3.5. If $Z(\mathbf{a}) = \varnothing$, we define $\Phi(\mathbf{a}) := \mathbf{a}$. Otherwise, if $Z(\mathbf{a}) = \{i_1, i_2, \ldots, i_k\} \neq \varnothing$ with $i_1 < i_2 < \cdots < i_k$, we define $\Phi(\mathbf{a})$ as follows. Let $\mathbf{b} := b_1 b_2 \ldots b_k = {t_k}^{-1}(a_{i_1} a_{i_2} \ldots a_{i_k}) \in \mathfrak{S}_k$ and $\sigma_{\mathbf{a}} \in \mathfrak{S}_n$ be the permutation of length n defined by

$$\sigma_{\mathbf{a}}(j) = \begin{cases} k+1, & \text{if } j = 1; \\ b_{j-1}, & \text{if } 2 \leq j \leq k+1; \\ j, & \text{if } k+2 \leq j \leq n. \end{cases}$$

and let

$$(\Phi(\mathbf{a}))(j) := \begin{cases} b_\ell, & \text{if } j = i_\ell \in Z(\mathbf{a}); \\ \sigma_{\mathbf{a}}(a_j), & \text{if } j \notin Z(\mathbf{a}). \end{cases}$$

If $\mathrm{Run}(\mathbf{a}) = \varnothing$, we define $\Psi(\mathbf{a}) := \mathbf{a}$. Otherwise, if $\mathrm{Run}(\mathbf{a}) = \{i_1, i_2, \dots, i_k\} \neq \varnothing$ with $i_1 < i_2 < \cdots < i_k$, we define $\Psi(\mathbf{a})$ as follows. Let $\mathbf{c} := c_1c_2 \dots c_k = t_k(a_{i_1}a_{i_2} \dots a_{i_k}) \in [1] \times [2] \times \cdots \times [k]$ and $\tau_{\mathbf{a}} \in \mathfrak{S}_n$ be the permutation of length n defined by

$$\tau_{\mathbf{a}}(j) = \begin{cases} \ell + 1, & \text{if } j = a_{i_\ell} \in [k]; \\ 1, & \text{if } j = k+1; \\ j, & \text{if } k+2 \leq j \leq n. \end{cases}$$

and let

$$(\Psi(\mathbf{a}))(j) := \begin{cases} c_\ell, & \text{if } j = i_\ell \in \mathrm{Run}(\mathbf{a}); \\ \tau_{\mathbf{a}}(a_j), & \text{if } j \notin \mathrm{Run}(\mathbf{a}). \end{cases}$$

Proof of Lemma 3.4.

(1) Let $Z(\mathbf{a}) = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$ and $\Phi(\mathbf{a}) =: \mathbf{d} = d_1 \cdots d_n$. On the one hand, $k \leq \mathrm{run}(\mathbf{d})$ since $\{d_{i_1}, \dots, d_{i_k}\} = [k]$. On the other hand, $k+1 \notin \{d_1, \dots, d_n\}$ because $k+1 \notin \{d_{i_1}, \dots, d_{i_k}\}$ and there is no $j \in [n] \setminus Z(\mathbf{a})$ such that $a_j = 1$. Hence $z(\mathbf{a}) = k = \mathrm{run}(\mathbf{d})$. Finally, let $i_j \in Z(\mathbf{a})$. Then $[d_{i_j}] \subseteq [k] \subseteq \{d_1, \dots, d_n\}$, which proves that $Z(\mathbf{a})$ is a subset of $\mathrm{Run}(\mathbf{d})$ with the same size.

Similarly, one can show that $\mathrm{run}(\mathbf{a}) = z(\Psi(\mathbf{a}))$ and $\mathrm{Run}(\mathbf{a}) = Z(\Psi(\mathbf{a}))$.

(2) Given $\mathbf{a} \in [n]^n$, we have $Z(\mathbf{a}) = \mathrm{Run}(\Phi(\mathbf{a}))$, $\mathrm{Run}(\mathbf{a}) = Z(\Psi(\mathbf{a}))$, $\tau_{\Phi(\mathbf{a})} = {\sigma_{\mathbf{a}}}^{-1}$ and $\sigma_{\Psi(\mathbf{a})} = {\tau_{\mathbf{a}}}^{-1}$. Hence $(\Psi \circ \Phi)(\mathbf{a}) = \mathbf{a} = (\Phi \circ \Psi)(\mathbf{a})$.

(3) Let $\mathbf{a} \in \mathsf{PF}_n$ and $k = z(\mathbf{a})$. If $j \leq k$, $|\Phi(\mathbf{a})^{-1}([j])| \geq j$, since $[j] \subseteq [k] \subseteq \Phi(\mathbf{a})([n])$. If $j > k$, then $\Phi(\mathbf{a})^{-1}([j]) = \mathbf{a}^{-1}([j])$ and so $|\Phi(\mathbf{a})^{-1}([j])| = |\mathbf{a}^{-1}([j])| \geq j$ because $\mathbf{a} \in \mathsf{PF}_n$. Since $\Phi(\mathsf{PF}_n) \subseteq \mathsf{PF}_n$ and Φ is a bijection, $\Phi(\mathsf{PF}_n) = \mathsf{PF}_n$. $\square$

Example 3.6. Let $\mathbf{a} = 341183414 \in [9]^9$. On the one hand, $Z(\mathbf{a}) = \{3, 4, 6, 7, 8, 9\}$, ${t_6}^{-1}(a_3a_4a_6a_7a_8a_9) = {t_6}^{-1}(113414) = 521634 \in \mathfrak{S}_6$, so $\sigma_{\mathbf{a}} = 7\underline{521634}89$ and $\Phi(\mathbf{a}) = 21\underline{52}8\underline{1634}$.

$$\mathbf{a} \quad = \quad \underline{3 \quad 4} \quad \underbrace{\boxed{1}}_{3} \quad \underbrace{\boxed{1}}_{4} \quad \underline{8} \quad \underbrace{\boxed{3}}_{6} \quad \underbrace{\boxed{4}}_{7} \quad \underbrace{\boxed{1}}_{8} \quad \underbrace{\boxed{4}}_{9}$$

$$\Phi(\mathbf{a}) \quad = \quad \underline{2 \quad 1} \quad \underbrace{\boxed{5}}_{3} \quad \underbrace{\boxed{2}}_{4} \quad \underline{8} \quad \underbrace{\boxed{1}}_{6} \quad \underbrace{\boxed{6}}_{7} \quad \underbrace{\boxed{3}}_{8} \quad \underbrace{\boxed{4}}_{9}$$

On the other hand, $\mathrm{Run}(\mathbf{a}) = \{8\}$, $\mathbf{c} = t_1(a_8) = t_1(1) = 1$, so $\tau_{\mathbf{a}} = 2\underline{1}3456789$ and $\Psi(\mathbf{a}) = 3422834\underline{1}4$. Note that $\mathbf{a}$ belongs to PF_9, as well as $\Phi(\mathbf{a})$ and $\Psi(\mathbf{a})$.

4. From parking functions to rook words

4.1. **Restricted integer sequences.** We start this section by considering a general situation of independent interest.

Definition 4.1. Let, for a positive integer k and for $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k$, $\mathbf{L} = (L_1, \dots, L_k) \in \mathbb{N}^k$ be the cumulative sum of $\boldsymbol{\ell}$, i.e.,

$$L_i = \ell_1 + \ell_2 + \cdots + \ell_i\,, \quad i = 1, \dots, k\,,$$

and consider the set

$$\langle \ell_1, \dots, \ell_k \rangle = \left\{ (x_0, x_1, \dots, x_k) \in \mathbb{Z}^{k+1} \mid x_0 = 0; \forall 1 \leq i \leq k\,, x_{i-1} < x_i \leq L_i \right\}$$

Lemma 4.2. *For all positive integers $k, \ell_1, \dots, \ell_k$, if $i < k$ and $\ell_{i+1} > 1$, then*

$$\begin{aligned}&\left|\langle \ell_1, \dots, \ell_{i-1}, \ell_i + 1, \ell_{i+1} - 1, \ell_{i+2}, \dots, \ell_k\rangle\right|\\ &\qquad = \left|\langle \ell_1, \dots, \ell_k\rangle\right| + \left|\langle \ell_1, \dots, \ell_{i-1}\rangle\right|\left|\langle \ell_{i+1} - 1, \ell_{i+2}, \dots, \ell_k\rangle\right|\end{aligned}$$

whereas

$$\left|\langle \ell_1, \dots, \ell_{k-1}, \ell_k + 1\rangle\right| = \left|\langle \ell_1, \dots, \ell_k\rangle\right| + \left|\langle \ell_1, \dots, \ell_{k-1}\rangle\right|$$

and, if $\ell_1 > 1$,

$$\left|\langle \ell_1 - 1, \ell_2, \dots, \ell_k\rangle\right| = \left|\langle \ell_1, \dots, \ell_k\rangle\right| - \left|\langle \ell_1 + \ell_2 - 1, \ell_3, \dots, \ell_k\rangle\right| .$$

Proof. We present here a bijective proof. Note that, for every $i < k$,

$$\langle \ell_1, \dots, \ell_k\rangle \subseteq \langle \ell_1, \dots, \ell_{i-1}, \ell_i + 1, \ell_{i+1} - 1, \ell_{i+2}, \dots, \ell_k\rangle \ .$$

But, by definition,

$$\begin{aligned}&(x_0, \dots, x_k) \in \langle \ell_1, \dots, \ell_{i-1}, \ell_i + 1, \ell_{i+1} - 1, \ell_{i+2}, \dots, \ell_k\rangle \setminus \langle \ell_1, \dots, \ell_k\rangle\\ &\iff \begin{cases} x_0 = 0 \\ x_i = L_i + 1 \\ x_{j-1} < x_j \le L_j \text{ for every } j \neq i \text{ with } 1 \le j \le k, \end{cases}\\ &\iff \begin{cases} (x_0, \dots, x_{i-1}) \in \langle \ell_1, \dots, \ell_{i-1}\rangle \\ (x_i - L_i - 1, \dots, x_k - L_i - 1) \in \langle \ell_{i+1} - 1, \ell_{i+2}, \dots, \ell_k\rangle \end{cases}\end{aligned}$$

For the second statement, note that also $\langle \ell_1, \dots, \ell_k\rangle \subseteq \langle \ell_1, \dots, \ell_{k-1}, \ell_k + 1\rangle$ and that $(x_0, \dots, x_k) \in \langle \ell_1, \dots, \ell_{k-1}, \ell_k + 1\rangle \setminus \langle \ell_1, \dots, \ell_k\rangle$ if and only if $x_k = L_k + 1$ and $(x_0, \dots, x_{k-1}) \in \langle \ell_1, \dots, \ell_{k-1}\rangle$.

Finally, for the third statement, note that, by definition, if

$$(0, x_1, \dots, x_k) \in \langle \ell_1 - 1, \ell_2, \dots, \ell_k\rangle \ ,$$

then

$$\begin{cases} x_1 + 1 > 1 \\ (0, x_1 + 1, \dots, x_k + 1) \in \langle \ell_1, \ell_2, \dots, \ell_k\rangle \ . \end{cases}$$

In fact, given a k-tuple $(y_1, \dots, y_k) \in \mathbb{N}^k$,

$$\begin{cases} (0, y_1, \dots, y_k) \in \langle \ell_1, \ell_2, \dots, \ell_k\rangle \\ (0, y_1 - 1, \dots, y_k - 1) \notin \langle \ell_1 - 1, \ell_2, \dots, \ell_k\rangle \end{cases}$$

if and only if

$$\begin{cases} y_1 = 1 \\ (0, y_2 - 1, \dots, y_k - 1) \in \langle \ell_1 + \ell_2 - 1, \ell_3, \dots, \ell_k\rangle . \end{cases} \quad \square$$

Remark 4.3. Let, for any $\mathbf{x} = (0, x_1, \dots, x_k) \in \mathbb{Z}^{k+1}$, $\mathbf{y} = (y_1, \dots, y_k) = (x_1 - 1, x_2 - 2, \dots, x_k - k)$. Then $\mathbf{x} \in \langle \ell_1, \dots, \ell_k\rangle$ if and only if $0 \le y_1 \le L_1 - 1$ and $y_i \le y_{i+1} \le L_{i+1} - (i + 1)$ for every $i = 1, 2, \dots, k - 1$.

Hence, if $(\ell_1, \dots, \ell_k)$ is a *composition of* n (i.e., $n = L_k$) we may represent the elements of $\langle \ell_1, \dots, \ell_k\rangle$ by lattice paths from $(0, 0)$ to $(k, n - k)$ that are contained in the region between the x axis and the path P that has the same ends and the property that the height of the ith horizontal step is $L_i - i$ for every $i = 1, 2, \dots, k$. See Figure 1 for an example. Hence,

$$\left|\langle \ell_1, \dots, \ell_k\rangle\right| = \det_{1 \le i,j \le k} \left(\binom{\ell_1 + \cdots + \ell_i - i + 1}{j - i + 1} \right) . \tag{4.3.1}$$

follows (cf. [8, Theorem 10.7.1]). Note that Lemma 4.2 may easily be proved by using the characteristic properties of determinants.

Definition 4.4. Given integers t, r and k such that $0 < r < k < n$, and a k-composition $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k$ of n, let

$$s(\boldsymbol{\ell}, t) := \sum_{i=0}^{k-1} \left| \left\langle (\textstyle\sum_{j=1}^{r} \ell_{i+j}) + t, \ell_{i+r+1}, \dots, \ell_{i+k-1} \right\rangle \right|,$$

where indices are to be read modulo k.

Example 4.5. Note that $(0,1,2)$ is a subsequence of $\mathbf{x} = (0,1,2,7,9)$ but $(0,1,2,3)$ is not. Let S be the set of elements of $\langle 3,1,5,2 \rangle$ with this property,

$$S = \{(0,1,2,x_3,x_4) \in \langle 3,1,5,2 \rangle \mid x_3 > 3\}$$

and note that the lattice paths associated with the elements of S are those which start by 2 horizontal steps, followed by a vertical step (cf. Figure 1). Now, $(0,1,2,x_3,x_4) \mapsto (0, x_3 - 3, x_4 - 3)$ defines a bijection between S and $\langle (3+1+5) - 3, 2 \rangle = \langle 6, 2 \rangle$.

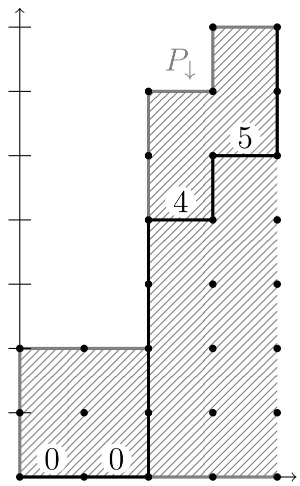


FIGURE 1. Lattice path representation of $(0,1,2,7,9) \in \langle 3,1,5,2 \rangle$.

Consider the 5-composition $(3,1,5,2,4)$ of 15, define similarly to S the four sets $T \subseteq \langle 1,5,2,4 \rangle$, $U \subseteq \langle 5,2,4,3 \rangle$, $V \subseteq \langle 2,4,3,1 \rangle$ and $W \subseteq \langle 4,3,1,5 \rangle$. Then $|S| + |T| + |U| + |V| + |W| = \left|\begin{smallmatrix} 6 & 15 \\ 1 & 7 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 5 & 10 \\ 1 & 8 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 8 & 28 \\ 1 & 10 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 6 & 15 \\ 1 & 6 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 5 & 10 \\ 1 & 9 \end{smallmatrix}\right| = 27 + 30 + 52 + 21 + 35 = 165$. A similar construction for another 5-composition c of 15 gives again this number. If, e.g., $c = (2,1,7,3,2)$, we obtain in the same manner $\left|\begin{smallmatrix} 7 & 21 \\ 1 & 9 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 8 & 28 \\ 1 & 9 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 9 & 36 \\ 1 & 10 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 4 & 6 \\ 1 & 4 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 2 & 1 \\ 1 & 8 \end{smallmatrix}\right| = 42 + 44 + 54 + 10 + 15 = 165$.

Theorem 4.6. *The value of $s(\boldsymbol{\ell}, t)$ does not depend on the k-composition $\boldsymbol{\ell}$.*

Proof. Note that, by definition, if we cyclically permute the elements of $\boldsymbol{\ell}$ the value of $s(\boldsymbol{\ell}, t)$ does not change. Hence, it is sufficient to prove that, given two k-compositions, $\mathbf{m} = (m_1, \dots, m_k)$ and $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$ such that

$$(m_1, m_2, \dots, m_{k-1}, m_k) = (\ell_1 - 1, \ell_2, \dots, \ell_{k-1}, \ell_k + 1),$$

we must have $s(\mathbf{m}, t) = s(\boldsymbol{\ell}, t)$.

Let $s_i(\boldsymbol{\ell}, t) = \left|\left\langle(\sum_{j=1}^{r} \ell_{i+j}) + t, \ell_{i+r+1}, \ldots, \ell_{i+k-1}\right\rangle\right|$ and define $s_i(\mathbf{m}, t)$ similarly. Then $s_0(\mathbf{m}, t) - s_0(\boldsymbol{\ell}, t) = \langle L_r + t - 1, \ell_{r+1}, \ldots, \ell_{k-1}\rangle - \langle L_r + t, \ell_{r+1}, \ldots, \ell_{k-1}\rangle$, which is the opposite of $\langle L_{r+1} + t - 1, \ell_{r+2}, \ldots, \ell_{k-1}\rangle$ by Lemma 4.2.

In general, by subtracting and subsequently applying Lemma 4.2 term by term, we obtain

$$\begin{aligned}
\sum_{i=0}^{k-1}(s_i(\mathbf{m}, t) - s_i(\boldsymbol{\ell}, t)) &= \sum_{i=0}^{k-r}(s_i(\mathbf{m}, t) - s_i(\boldsymbol{\ell}, t)) \\
&= - \left|\left\langle L_{r+1} + t - 1, \ell_{r+2}, \ldots, \ell_{k-1}\right\rangle\right| \\
&\quad + \left|\left\langle(\textstyle\sum_{j=1}^{r} \ell_{j+1}) + t, \ell_{r+2}, \ldots, \ell_{k-1}\right\rangle\right| \\
&\quad + \left|\left\langle(\textstyle\sum_{j=1}^{r} \ell_{j+2}) + t, \ell_{r+3}, \ldots, \ell_{k-1}\right\rangle\right| \left|\left\langle \ell_1 - 1\right\rangle\right| \\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
&\quad + \left|\left\langle(\textstyle\sum_{j=1}^{r} \ell_{j+k-r-1}) + t\right\rangle\right| \left|\left\langle \ell_1 - 1, \ell_2, \ldots, \ell_{k-r-2}\right\rangle\right| \\
&\quad + \left|\left\langle \ell_1 - 1, \ell_2, \ldots, \ell_{k-r-1}\right\rangle\right|
\end{aligned}$$

We prove that this number is zero by proving that the opposite of the first summand, the size of $\mathfrak{X} = \langle L_{r+1} + t - 1, \ell_{r+2}, \ldots, \ell_{k-1}\rangle$, is the sum of the other summands, each of which counts the elements with the same image by the function $f \colon \mathfrak{X} \to [k - r]$ such that

$$f(0, x_1, \ldots, x_{k-r-1}) = \begin{cases} k - r, & \text{if } x_i < L_i,\ \forall\, i \leq k - r - 1; \\ \min\{i \mid x_i \geq L_i\}, & \text{otherwise.} \end{cases}$$

First, note that $f(X) = k - r$ if and only if $X \in \langle \ell_1 - 1, \ell_2, \ldots, \ell_{k-r-1}\rangle$. If $X \notin \langle \ell_1 - 1, \ell_2, \ldots, \ell_{k-r-1}\rangle$, then

$$\begin{aligned}
f(X) \geq i &\iff \min\{t \mid x_t \geq L_t\} \geq i \\
&\iff \forall_{j<i}\,,\ x_j < L_j
\end{aligned}$$

and hence

$$f(X) = i \iff (\forall_{j<i}\,,\ x_j < L_j) \wedge x_i \geq L_i\,.$$

Finally,

$$(0, x_1, \ldots, x_{k-r-1}) \mapsto \Big((0, x_1, \ldots, x_{i-1}), (0, x_i - L_i + 1, \ldots, x_{k-r-1} - L_i + 1)\Big)$$

defines a bijection between $f^{-1}(\{i\}) \subseteq \mathfrak{X}$ and the set

$$\langle \ell_1 - 1, \ell_2, \ldots, \ell_{i-1}\rangle \times \left\langle(\textstyle\sum_{j=1}^{r} \ell_{j+i}) + t, \ell_{r+i+1}, \ldots, \ell_{k-1}\right\rangle. \quad \square$$

4.2. **Counting parking functions and rook words with a given type.** Recall that a parking function of length n is a tuple $\mathbf{a} = (a_1, \ldots, a_n) \in [n]^n$ such that the ith entry *in ascending order* is always at most $i \in [n]$. In other words,

$$\mathbf{a} \in \mathsf{PF}_n \text{ if, for every } i \in [n],\ \left|\mathbf{a}^{-1}([i])\right| \geq i\,.$$

Definition 4.7. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and suppose that $\{a_1, \ldots, a_n\} = \{x_1, \ldots, x_k\}$ with $x_i < x_j$ whenever $1 \leq i < j \leq k$.

The *reduced image of* $\mathbf{a}$ is

$$\mathrm{rim}(\mathbf{a}) = (x_1 - 1, \ldots, x_k - 1) \in (\mathbb{N} \cup \{0\})^k\,;$$

the *coimage of* $\mathbf{a}$ is the quotient set

$$\operatorname{coim}(\mathbf{a}) = \{\bar{x} \mid x \in [n]\},$$

where $\bar{x} := \mathbf{a}^{-1}(\mathbf{a}(x))$, ordered by

$$\bar{x} < \bar{y} \iff \mathbf{a}(x) < \mathbf{a}(y)\,.$$

Let $\mathfrak{A} = (A_1, \ldots, A_k)$ be an ordered (set) partition of $[n]$. The *length-vector* of $\mathfrak{A}$ is

$$\boldsymbol{\ell}(\mathfrak{A}) = (|A_1|, \ldots, |A_{k-1}|)\,.$$

Lemma 4.8. *Let $\mathfrak{A}$ be an ordered partition of $[n]$ with length-vector $\boldsymbol{\ell}(\mathfrak{A}) = (\ell_1, \ldots, \ell_{k-1})$ and let $\mathbf{a}\colon [n] \to [n]$ be such that $\operatorname{coim}(\mathbf{a}) = \mathfrak{A}$.*
Then $\mathbf{a}$ is a parking function *if and only if*

$$\operatorname{rim}(\mathbf{a}) \in \langle \ell_1, \ldots, \ell_{k-1} \rangle$$

and $\mathbf{a}$ is a run r parking function *if and only if*

$$\begin{cases} \operatorname{rim}(\mathbf{a}) = (0, 1, \ldots, r-1, x_{r+1}, \ldots, x_k) \\ (0, x_{r+1} - r, \ldots, x_k - r) \in \langle (\sum_{i=1}^{r} \ell_i) - r, \ell_{r+1}, \ldots, \ell_{k-1} \rangle \end{cases}$$

Proof. Follows immediately from the definitions. □

Note that $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{RW}_n$ if and only if, for $i = a_1 - 1$, $\operatorname{rim}(\mathbf{a})$ belongs to the set

$$\Big\langle \underbrace{1, \ldots, 1}_{i \text{ times}}, n-k+1, \underbrace{1, \ldots, 1}_{k-i-2 \text{ times}} \Big\rangle =$$
$$= \big\{(0, 1, \ldots, i, x_{i+1}, \ldots, x_{k-1}) \in [n-1]^k \mid i < x_{i+1} < \cdots < x_{k-1} \leq n-1\big\}\,.$$

Hence, if we denote by PF_n^r the set of run r parking functions of length n, for $\mathfrak{A} = (A_1, \ldots, A_k)$, according to (4.3.1) and by definition

$$\left|\mathsf{PF}_n \cap \operatorname{coim}^{-1}(\mathfrak{A})\right| = \det_{1 \leq i,j \leq k-1} \left(\binom{|A_1| + \cdots + |A_i| - i + 1}{j - i + 1} \right)$$

$$\left|\mathsf{PF}_n^r \cap \operatorname{coim}^{-1}(\mathfrak{A})\right| = \det_{r \leq i,j \leq k-1} \left(\binom{|A_1| + \cdots + |A_i| - i}{j - i + 1} \right)$$

$$\left|\mathsf{RW}_n \cap \operatorname{coim}^{-1}(\mathfrak{A})\right| = \binom{n-1-i}{k-1-i}$$

Definition 4.9. Given an ordered partition $\mathfrak{A}$ of $[n]$ and $\mathbf{a} \in [n]^n$, we say that $\mathbf{a}$ *is of type* $\mathfrak{A}$ if the coimage of $\mathbf{a}$ is a *cyclic permutation of* $\mathfrak{A}$.

We denote the set of type $\mathfrak{A}$ elements of $[n]^n$ by

$$\begin{aligned} \overline{\operatorname{coim}^{-1}}(\mathfrak{A}) = {} & \operatorname{coim}^{-1}(A_1, A_2, \ldots, A_k) \cup \\ & \operatorname{coim}^{-1}(A_2, A_3, \ldots, A_1) \cup \cdots \cup \\ & \operatorname{coim}^{-1}(A_k, A_1, \ldots, A_{k-1})\,. \end{aligned}$$

Theorem 4.10. *Let $\mathfrak{A} = (A_1, \ldots, A_k)$ be an ordered partition of $[n]$ and let $1 \leq r \leq n$ for a natural number n. Then*

- *the number of* ***parking functions of type*** $\mathfrak{A}$*, as well as the number of* ***rook words of type*** $\mathfrak{A}$*, is* $\dbinom{n}{k-1}$*;*

- *the number of* ***run*** r ***parking functions of type*** $\mathfrak{A}$*, as well as the number of* ***run*** r ***rook words of type*** $\mathfrak{A}$*, is* $r\binom{n-r-1}{k-r}$.

Proof. Consider the two ordered k-compositions of $[n]$, $\mathcal{C} = (|A_1|, |A_2|, \dots, |A_k|)$ and $\mathcal{D} = (n-k+1, 1, \dots, 1)$, and apply Theorem 4.6 with different values of r and t.

For the first statement, take $r = 1$ and $t = 0$; in the notation thereof,

$$\left|\mathsf{PF}_n \cap \overline{\mathrm{coim}^{-1}}(\mathfrak{A})\right| = s(\mathcal{C}) \;=\; \sum_{i=0}^{k-1}\binom{n-1-i}{k-1-i} \;=\; s(\mathcal{D}) = \left|\mathsf{RW}_n \cap \overline{\mathrm{coim}^{-1}}(\mathfrak{A})\right|.$$

For the second statement, by taking $r = 1, \dots, k$ and $t = -r$ we obtain that

$$\left|\mathsf{PF}_n^r \cap \overline{\mathrm{coim}^{-1}}(\mathfrak{A})\right| = s(\mathcal{D}) = r\big|\langle n-k, \underbrace{1, \dots, 1}_{k-r-1 \text{ times}}\rangle\big| + 0\,,$$

since, for $\ell_1 = n-k+1$ and $\ell_2 = \cdots = \ell_k = 1$,

$$\left|\langle \textstyle\sum_{j=1}^{r} \ell_{i+j} - r, \ell_{i+r+1}, \dots, \ell_{i+k-1}\rangle\right| = \begin{cases} \left|\langle n-k, 1, \dots, 1\rangle\right|, & \text{if } i = 0 \text{ or } k-i \in [r-1]; \\ 0, & \text{otherwise.} \end{cases}$$

This shows that the number of run r parking functions of type $\mathfrak{A}$ is $r\binom{n-r-1}{k-r}$, since

$$\langle n-k, 1, \dots, 1\rangle = \left\{(x_1, \dots, x_{k-r}) \mid 0 < x_1 < \cdots < x_{k-r} \le n-r-1\right\}.$$

Finally, note that, for example, all the type $\mathfrak{A}$ elements $\mathbf{a} = (a_1, a_2, \dots, a_n) \in [n]^n$ with $a_1 = 1$ share the same coimage, and that there are $\binom{n-r-1}{k-r}$ such rook words with run r, since they are determined by the last $k-r$ strictly increasing coordinates of $\mathrm{rim}(\mathbf{a})$, all of them greater than r and less than n. The same happens if $a_1 = i$ for $1 \le i \le r$, and a_1 cannot be greater than r, by definition. □

We note that the first part of Theorem 4.10 can be obtained directly from [10, Cyclic Lemma], where the following bijection is defined. Let $\mathbf{b} = \mathbf{a}$ if $\mathbf{a} \in \mathsf{PF}_n \cap \mathsf{RW}_n$ and, if $\mathbf{a} \in \mathsf{PF}_n \setminus \mathsf{RW}_n$ and $m = \max\big([a_1] \setminus \mathbf{a}([n])\big)$, let $\mathbf{b} = (b_1, \dots, b_n) \in [n]^n$ be such that

$$a_i \equiv b_i + m \pmod n\,;$$

then $\mathbf{a} \mapsto \mathbf{b}$ defines a bijection between the set of parking functions and the set of rook words *of a given type*.

Theorem 4.11. *For every* $n \in \mathbb{N}$,

$$\mathcal{RPF}_n(t) = \mathcal{RRW}_n(t)\,.$$

Proof. Follows immediately from Theorem 4.10. □

5. Counting rook words with a given run

Given positive integers n and r such that $r \le n$, let

$$\mathsf{RW}_n^r = \{f \in \mathsf{RW}_n \mid \mathrm{run}(f) = r\}$$

and for $\mathbf{a} = (a_1, \dots, a_n) \in \mathsf{RW}_n^r$ let

$$\mathbf{r} = \mathbf{r}(\mathbf{a}) = (i_1, \dots, i_r)$$

where $i_j = \min\{i \in [n] \mid a_i = j\}$ for every $j \in [r]$ (compare with the definition of Run in page 6).

Theorem 5.1. *For all integers $1 \leq r \leq n$,*

$$[t^r]\big(\mathcal{ALT}_n(t)\big) = r! \sum_{e_1+\cdots+e_r=n-r} (n-1)^{e_1}(n-2)^{e_2}\cdots(n-r)^{e_r} \tag{5.1.1}$$

$$= r \sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}(n-1-j)^{n-1}\,. \tag{5.1.2}$$

Proof. We have seen before that

$$[t^r]\big(\mathcal{ALT}_n(t)\big) = [t^r]\big(\mathcal{RRW}_n(t)\big) = \big|\mathsf{RW}_n^r\big|.$$

Given $\mathbf{a} \in \mathsf{RW}_n^r$ and $\pi \in \mathfrak{S}_r$, let $\pi\mathbf{a}$ be the element of $[n]^n$ defined by

$$(\pi\mathbf{a})(j) = \begin{cases} \pi(a_j) & \text{if } a_j \leq r\,; \\ a_j & \text{if } a_j > r\,. \end{cases}$$

Note that $\pi\mathbf{a} \in \mathsf{RW}_n^r$ if and only if $\mathbf{a} \in \mathsf{RW}_n^r$. Owing to this, the left-hand side of (5.1.1) is equal to $r!$ times the number of elements of

$$A = \{\mathbf{a} \in \mathsf{RW}_n^r \mid \mathbf{r}(\mathbf{a}) = (i_1,\ldots,i_r) \text{ with } 1 = i_1 < i_2 < \cdots < i_r\}.$$

Now, for a fixed sequence $1 = i_1 < \cdots < i_r$, $\mathbf{a} = (a_1,\ldots,a_n) \in A$ with $\mathbf{r}(\mathbf{a}) = (i_1,\ldots,i_r)$ if and only if, for every $1 \leq j \leq r$,

- $a_{i_j} = j$,

and for every $1 \leq \ell \leq n$,

- $a_\ell \notin \{j+1,\ldots,r,r+1\}$, if $i_j < \ell < i_{j+1}$ for some $j \in [r-1]$;
- $a_\ell \neq r+1$, if $\ell > i_r$.

This gives (5.1.1) for $e_j = i_{r+2-j} - i_{r+1-j} - 1$ with $1 < j \leq r$, and $e_1 = n - i_r$.

We note that the right-hand side of (5.1.2) is, by the Inclusion-Exclusion Principle, r times the number of elements of

$$B = \big\{f \colon [n-1] \to [n-1] \mid [r-1] \subseteq f([n-1])\big\}\,.$$

Given $\ell \in [n]$ with $\ell \leq r < n$, consider the bijection $\varphi_\ell \colon [n] \setminus \{r+1\} \to [n-1]$ such that

$$\varphi_\ell(j) = \begin{cases} j, & \text{if } j < \ell\,; \\ r, & \text{if } j = \ell\,; \\ j-1, & \text{if } j > \ell\,. \end{cases}$$

and note that $[r-1] \subseteq \varphi_\ell([r])$. Now, $F(a_1,\ldots,a_n) = (a_1,\varphi_{a_1}(a_2),\ldots,\varphi_{a_1}(a_n))$ clearly defines a bijection from RW_n^r to $[r] \times B$. □

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